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# DEVELOPMENT OF THERMOCAPILLARY CONVECTION IN A FLUID CYLINDER AND CYLINDRICAL AND PLANE LAYERS UNDER THE INFLUENCE OF INTERNAL HEAT SOURCES

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Under weightless conditions, neither external forces nor forces associated with self-gravitation are strong enough to cause convective motion. However, convection may develop due to the fact that surface tension is dependent on temperature.

The studies [1-4] investigated the conditions for the development of convection in a fluid during the heating of a solid or free surface. Here, we study the stability of the equilibrium state which develops in a liquid cylinder and cylindrical and plane layers under the influence of constant internal heat sources. Explicit formulas are obtained for the critical Marangoni numbers. It is shown that allowance for deformation of the free surface leads to a decrease in stability and the appearance of a discontinuity on the neutral curve. Also, the equilibrium state of the plane layer is more stable than in the analogous Pearson problem [1].

1. Fluid Cylinder. Let a quiescent fluid cylinder contain constant internal heat sources of intensity q. Then the equilibrium state is described by the formulas

$$u = v = w = 0,\ p = \mathrm{const},\ \Theta(r) = -qr^2/(4\chi) + \mathrm{const}. \qquad (1.1)$$

Here, (u, v, w) are components of the velocity vector in the cylindrical coordinate system (r, φ, z); p is pressure; Θ is temperature; χ = const is the diffusivity of the fluid.

As the characteristic scales of length, velocity, pressure, and temperature, we choose the quantities b, $\nu/b$, $\rho\nu^2/b^2$, and $\nu\gamma b/\chi$ (b is the radius of the cylinder, ν is kinematic

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viscosity, $\rho$ is the density of the fluid, $\gamma = -\Theta_r(b) = qb/2\chi$). Equations describing small perturbations of a random thermocapillary motion in cylindrical coordinates were derived in [5]. Assuming these perturbations to be dependent on $\varphi$, z, and t in accordance with the law $\exp[i(m\varphi + \alpha z - \alpha iCt)]$, we obtain amplitude equations for the equilibrium state (1.1):

$$\mu U + P' = \left[\frac{1}{\xi}(\xi U)'\right]' - \frac{2im}{\xi^2}V; \tag{1.2}$$

$$\mu V + \frac{im}{\xi}P = \left[\frac{1}{\xi}(\xi V)'\right]' + \frac{2im}{\xi^2}U; \tag{1.3}$$

$$\mu W + i\alpha P = \frac{1}{\xi}(\xi W')'; \tag{1.4}$$

$$(\xi U)' + imV + i\alpha\xi W = 0; \tag{1.5}$$

$$\mu T - \xi U = \frac{1}{\xi}(\xi T')' \quad (0 < \xi < 1); \tag{1.6}$$

$$V' - \frac{V}{\xi} + \frac{im}{\xi}U = -\frac{im}{\xi}\mathrm{Ma}(T + \Theta_0' R); \tag{1.7}$$

$$i\alpha U + W' = -i\alpha\,\mathrm{Ma}(T + \Theta_0' R); \tag{1.8}$$

$$-i\alpha CR = U; \tag{1.9}$$

$$-P + 2U' = \mathrm{We}(1 - \alpha^2 - m^2)R - \mathrm{Ma}(T + \Theta_0' R); \tag{1.10}$$

$$T' + \mathrm{Bi}\,T + (\Theta_0'' + \mathrm{Bi}\,\Theta_0')R = 0 \quad (\xi = 1); \tag{1.11}$$

$$|U|,\ |V|,\ |W|,\ |P|,\ |T| < \infty \ (\xi = 0), \tag{1.12}$$

where U, V, W, P, and T are perturbations of the components of the velocity vector, pressure, and temperature; R is the deviation of the boundary from its unperturbed state with respect to a normal; $r = b$; $\xi = r/b$; $\Theta_0 = \chi\Theta/\nu\gamma b$; $\mu = \alpha^2 + m^2/\xi^2 - i\alpha C$; $\mathrm{Ma} = \gamma\kappa b^2/\rho\nu\chi$ is the Marangoni number; $\kappa = -d\sigma/d\Theta = \mathrm{const} > 0$ is the temperature coefficient of surface tension, so that $\sigma = \sigma_0 - \kappa(\Theta - \Theta(b))$; $\mathrm{We} = b\sigma_0/\rho\nu\chi$ is the Weber number; $\alpha$ is the wave number along the z axis; m is the spectral mode with respect to the angle $\varphi$; C is a complex parameter; Bi is the Biot number; $\Theta_0'(1) = \Theta_0''(1) = -1$; a prime denotes differentiation with respect to $\xi$.

We adopt the principle of monotonic change of the perturbations, so that the boundary of stability is determined by values of C = 0 in (1.2)-(1.4), (1.6), (1.9). The condition of the existence of a nontrivial solution of the problem makes it possible to find a critical value Ma($\alpha$, m, We, Bi) at which the equilibrium becomes unstable.

Let us first examine an axisymmetric perturbation (m = 0). The problem for the function V is separate and does not contain Ma. After we exclude W and P, we obtain the following problem for the function U:

$$L^2U = 0 \quad (0 < \xi < 1),\ L = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - \left(\alpha^2 + \frac{1}{\xi^2}\right); \tag{1.13}$$

$$U(1) = 0,\ U''(1) + U'(1) = -\alpha^2\,\mathrm{Ma}\,[T(1) - R]; \tag{1.14}$$

$$U'''(1) + 2U''(1) - (1 - 3\alpha^2)U'(1) + \alpha^2\,\mathrm{We}(1 - \alpha^2)R + \alpha^2\,\mathrm{Ma}\,[T(1) - R] = 0; \tag{1.15}$$

$$U(0) < \infty,\ (\xi U)'/\xi < \infty. \tag{1.16}$$

We introduce the function $\psi(\xi)$, so that

$$L^2\psi = 0\ (0 < \xi < 1),\ \psi(0) < \infty,\ (\xi\psi)'/\xi < \infty,$$
$$\psi(1) = \psi''(1) + \psi'(1) - 1 = 0.$$

The solution of the last problem has the form

$$\psi(\xi) = -\frac{I_2(\alpha)}{2\alpha I_1^2(\alpha)}I_1(\alpha\xi) + \frac{\xi I_2(\alpha\xi)}{2\alpha I_1(\alpha)} \tag{1.17}$$

[$I_{1,2}(x)$ are modified Bessel functions of the first kind]. We find from (1.13), (1.14), and (1.16) that $U = -\alpha^2\mathrm{Ma}[T(1) - R]\psi(\xi)$. The perturbation of temperature T is found from the solution of boundary-value problem (1.6), (1.11), (1.12), and $T(1) = f(\alpha)R$ with a known function $f(\alpha)$. Using these relations and the properties of Bessel functions, we obtain the following representation from boundary condition (1.15)

$$\mathrm{Ma} = \frac{(1-\alpha^2)(\alpha s + \mathrm{Bi})}{(\alpha^2-1)F(\alpha, s) + G(\alpha, s)\,\mathrm{We}^{-1}\mathrm{Pr}^{-1}}, \tag{1.18}$$

where

$$s = \frac{I_1(\alpha)}{I_0(\alpha)};\ F = \frac{\alpha}{12}\left(\frac{1}{s} - s\right) - \frac{1}{6\alpha}s - \frac{1}{12};\ G = \frac{2}{s^2}(1-\alpha s)(\alpha^2 - \alpha^2 s^2 + s^2 - \alpha s).$$

For azimuthal perturbations, $\alpha = 0$, and the problem for W is separated. It is easy to see that $W \equiv 0$. The pressure perturbation satisfies the equation $P'' + P'/\xi + m^2P/\xi^2 = 0$, following from (1.2), (1.3), and (1.5). With allowance for the condition $P(0) < \infty$, the function $P = C_1\xi^m$ ($C_1$ = const) and Eqs. (1.2), (1.3) take the form $U'' + U'/\xi - (m^2 + 1)U/\xi^2 - 2imV/\xi^2 = C_1 m\xi^{m-1}$, $V'' + V'/\xi - (m^2 + 1)V/\xi^2 + 2imU/\xi^2 = C_1 im\xi^{m-1}$. The solution of this system is written as

$$U = \frac{1}{4}(C_1 + 2C_2)\xi^{m+1} + \frac{1}{2}C_3\xi^{m-1},$$
$$V = -\frac{i}{4}(2C_2 - C_1)\xi^{m+1} + \frac{i}{2}C_3\xi^{m-1},\ C_2, C_3 = \text{const}.$$

For the temperature perturbation, we find

$$T = C_4\xi^m - \frac{C_1 + 2C_2}{32m + 64}\xi^{m+4} - \frac{C_3}{8m+8}\xi^{m+2},\ C_4 = \text{const}.$$

Continuity equation (1.5) leads us to the relationship between the constants $C_1$ and $C_2$: $C_1 = -2(m + 1)C_2$. Substitution of the explicit expressions for U, V, P, and T into boundary conditions (1.7), (1.9)-(1.11) makes it possible to determine the critical value of Ma:

$$\mathrm{Ma}(m) = \frac{8(m+1)(m+2)(m+\mathrm{Bi})}{m + 16(m+2)\,\mathrm{We}^{-1}} \quad (m \neq 1). \tag{1.19}$$

It is clear that

$$\min_m \mathrm{Ma}(m) = \mathrm{Ma}(2) = \frac{48(\mathrm{Bi}+2)}{1 + 32\,\mathrm{We}^{-1}}. \tag{1.20}$$

At $m = 1$ and $\alpha = 0$, the solution of problem (1.2)-(1.12) has the form $W = 0$, $U = V = P = 0$, $T = C_4\xi$, $R = C_4$, which corresponds to displacement of the free surface without deformation in the plane z = const. If we immediately assume the surface to be undeformable, then this problem has a nontrivial solution and $\mathrm{Ma} = 48(1 + \mathrm{Bi})$. It is obtained formally from (1.19) with $m = 1$ and $\mathrm{We} = \infty$.

In the general case, we can use continuity equation (1.5) to exclude the function W. We will seek the solution of the resulting problem in the form

$$U = \gamma\varphi(\xi),\ V = \gamma\psi(\xi),\ P = \gamma f(\xi) \tag{1.21}$$

($\gamma = \mathrm{Ma}[T(1) + \Theta_0'(1)R]$). The function $f = C_1 I_m(\alpha\xi)$ ($C_1$ = const), while $\varphi(\xi)$ and $\psi(\xi)$ satisfy the system

$$\varphi'' + \frac{1}{\xi}\varphi' - \left(\alpha^2 + \frac{m^2+1}{\xi^2}\right)\varphi - \frac{2im}{\xi^2}\psi = C_1 I_m'(\alpha\xi),$$
$$\psi'' + \frac{1}{\xi}\psi' - \left(\alpha^2 + \frac{m^2+1}{\xi^2}\right)\psi + \frac{2im}{\xi^2}\varphi = \frac{im}{\xi}C_1 I_m(\alpha\xi),$$

which was solved in [6]. The solution found there can be written in the simpler form

$$\varphi = \frac{C_2}{2}I_{m+1}(\alpha\xi) + \frac{C_3}{2}I_{m-1}(\alpha\xi) + \frac{C_1}{2}\xi I_m(\alpha\xi),$$
$$\psi = -\frac{iC_2}{2}I_{m+1}(\alpha\xi) + \frac{iC_3}{2}I_{m-1}(\alpha\xi) \tag{1.22}$$

with the constants $C_2$, $C_3$.

The temperature perturbation is found from (1.6)

$$T = C_4 I_m(\alpha\xi) - \int_0^{\xi} \tau^2 U(\tau) D(\alpha\xi, \alpha\tau)\, d\tau, \tag{1.23}$$

where $D = I_m(\alpha\xi)K_m(\alpha\tau) - K_m(\alpha\xi)I_m(\alpha\tau)$; $K_m$ is a modified Bessel function of the second kind; $C_4$ = const.

After (1.21) and (1.23) are inserted into boundary conditions (1.10) and (1.11)

$$\mathrm{Ma}(\alpha, m) = \frac{(1-\alpha^2-m^2)\left[\alpha I'_m(\alpha) + \mathrm{Bi}\, I_m(\alpha)\right]}{(1-\alpha^2-m^2)F_m + G_m \mathrm{We}^{-1}} \tag{1.24}$$

$$\left(F_m = \int_0^1 \tau^2 \varphi(\tau) I_m(\alpha\tau)\, d\tau,\ G_m = \left[I_m(\alpha) - \alpha I'_m(\alpha)\right]\left[2\varphi'(1) - C_1 I_m(\alpha) + 1\right]\right).$$

The constants $C_1$, $C_2$, $C_3$ are determined from boundary conditions (1.7)-(1.9), which with allowance for (1.21) are rewritten in the form $\varphi(1) = 0$, $\varphi''(1) + \varphi'(1) = -\alpha^2 - m^2$, $\psi'(1) - \psi(1) = -im$.

It should be noted that $\lim\limits_{\alpha\to 0} \mathrm{Ma}(\alpha, m) = \mathrm{Ma}(m)$ where Ma(m) is determined from (1.19). We performed numerical calculations using Eqs. (1.18) and (1.24). In the latter case, m = 1, the integral is calculated explicitly through the Bessel functions $I_0(\alpha)$, $I_1(\alpha)$. These formulas are rather awkward and are not presented here.

Figure 1 shows the dependence of Ma on the wave number $\alpha$ for axisymmetric perturbations (m = 0). Curves 2 and 3 correspond to the case of an undeformed free boundary, when We = ∞. The minimum critical Marangoni number $\mathrm{Ma}_c = 48$ for Bi = 0 (curve 3) and $\alpha_c = 0$. It is interesting to note that, in accordance with (1.20), the minimum value of $\mathrm{Ma}_c$ for azimuthal perturbations is also equal to 48 for the same parameter values. If Bi = 2 (curve 2), then $\mathrm{Ma}_c = 178.7$ and $\alpha_c = 2.1$. The minimum value of $\mathrm{Ma}_c$ increases with an increase in heat transfer.

If We ≠ ∞, then the denominator in Eq. (1.18) may vanish at a certain $\alpha^*$. In particular, for $\mathrm{We} = 10^4$, the graph of Ma(α) has a vertical asymptote at $\alpha^* = 1.006$, $\mathrm{Ma}(\alpha) \le 0$ at $\alpha \in [1, \alpha^*)$, and $\mathrm{Ma}(\alpha) \to -\infty$ at $\alpha \to \alpha^* - 0$. For Bi = 0 (curve 4), the function Ma(α) reaches its positive maximum 48.8 in the interval $\alpha > \alpha^*$ at α = 0.8, while it reaches its positive minimum $\mathrm{Ma}_c = 57.7$ for $\alpha > \alpha^*$ at $\alpha_c = 1.2$. If Bi = 2 (curve 1), then $\mathrm{Ma}_c = 178.7$ at $\alpha_c = 2.1$.

It should be noted that the curves of Ma(α) merge for We → ∞. Also, Ma(1) = 0 at all We ≠ ∞ and $\mathrm{Ma}(\alpha) \sim 8\alpha(\alpha + \mathrm{Bi})$ at large α and We = ∞. The same asymptote is obtained exactly for azimuthal perturbations. This follows from Eqs. (1.19) at m → ∞ (We = ∞). Also valid is the analogous asymptote for the equilibrium of a plane layer bounded by free surfaces [3].

In the case m = 1, all of the curves Ma(α, 1) establishing the boundary of stability have a minimum. Discontinuities are absent, and the graphs for We = ∞ and $10^4$ are nearly indistinguishable. Curve 2 in Fig. 2 corresponds to a thermally-insulated boundary Bi = 0, and the minimum value is equal to 47.38 at α = 0.66, Ma(0, 1) = 48. With an increase in heat transfer, the minimum is shifted in the shortwave direction and is equal to 124.1

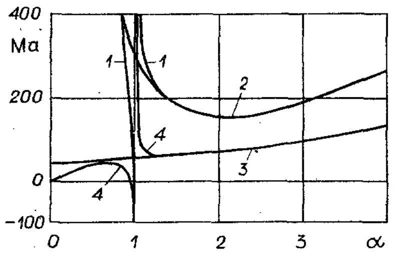


Fig. 1

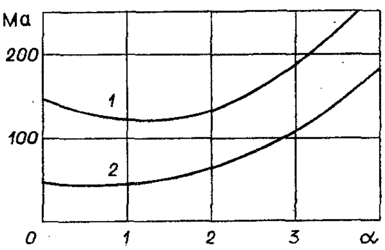


Fig. 2

for Bi = 2 (curve 1), where $\alpha = 1.27$ and Ma(0, 1) = 144. The curves Ma($\alpha$, m) for fixed $m \geq 2$ and We = $\infty$ are located above the curves Ma($\alpha$, 1). Thus, comparing Figs. 1 and 2, we conclude that the most dangerous perturbations are those for which m = 1.

2. Cylindrical Layer of Fluid. We will assume that the fluid is located on a solid cylindrical surface $r = a$ and occupies the region $a \leq r \leq b$, $0 \leq \varphi \leq 2\pi$, $-\infty < z < \infty$. We will write the equilibrium state, satisfying the condition of thermal insulation on the interior of the cylinder $0 \leq r \leq a$, in the form

$$u = v = w = 0,\ p = \text{const},\ \Theta(r) = \frac{qa^2}{2\chi}\ln\frac{r}{b} - \frac{q}{4\chi}r^2 + \text{const}. \tag{2.1}$$

In problem (1.2)-(1.12) concerning small perturbations, only Eq. (1.6) changes:

$$\mu T + \left(\frac{d^2}{\xi} - \xi\right) U = \frac{1}{\xi}(\xi T')' \quad (d < \xi < 1), \tag{2.2}$$

where $d = a/b < 1$, while boundedness conditions (1.12) are replaced by conditions of adhesion and thermal insulation on the inside of the cylinder

$$U = V = W = T' = 0 \ (\xi = d). \tag{2.3}$$

Also, it is necessary to put $\Theta_0'(1) = d^2 - 1$, $\Theta_0''(1) = -(d^2 + 1)$ in boundary conditions (1.7), (1.8), (1.10), and (1.11).

Here, for axisymmetric perturbations, instead of (1.17) we have

$$\psi = C_1\frac{\xi}{2\alpha}I_0(\alpha\xi) - C_2\frac{\xi}{2\alpha}K_0(\alpha\xi) + C_3I_1(\alpha\xi) + C_4K_1(\alpha\xi) \tag{2.4}$$

[$K_j(x)$ are modified Bessel functions of the second kind].

The constants $C_1, \ldots, C_4$ are found from conditions (1.8), (1.9), (2.3)

$$\psi(d) = 0,\ \psi'(d) = 0,\ \psi(1) = 0,\ \psi''(1) + \psi'(1) - 1 = 0. \tag{2.5}$$

Determining the perturbation of temperature T from (1.11), (2.2), and (2.3) and inserting the result into boundary condition (1.10), we find

$$\text{Ma} = \frac{(1-\alpha^2)(\alpha l_1 + \text{Bi}\, l_2)}{\alpha^3(1-\alpha^2)G(\alpha, d) + F(\alpha, d)\,\text{We}^{-1}}, \tag{2.6}$$

where

$$l_1 = I_1(\alpha)K_1(\alpha d) - I_1(\alpha d)K_1(\alpha); \qquad l_2 = I_1(\alpha d)K_0(\alpha) + I_0(\alpha)K_1(\alpha d);$$

$$G(\alpha, d) = \int_d^1 (d^2 - \tau^2)\psi(\tau)[K_0(\alpha\tau)I_1(\alpha d) + K_1(\alpha d)I_0(\alpha\tau)]\,d\tau; \qquad F(\alpha, d) =$$

$$= [\psi'''(1) - 3(1 + \alpha^2)\psi'(1) + 3]\,[(1 + d^2)l_2 + \alpha(d^2 - 1)l_1].$$

The integral G($\alpha$, d) can be expressed explicitly in terms of modified Bessel functions. It is awkward and is not presented here. It can be shown that at $d \to 0$, Eq. (2.6) becomes (1.18). For azimuthal perturbations, we seek the functions U, V, and P in the form (1.21), where $\varphi = (1/2)[-mC_1\xi^{m+1} + C_2\xi^{-m-1} + C_3\xi^{m-1} + mC_4\xi^{-m+1}]$, $\psi = -(i/2)\ [(m + 2) \times C_1\xi^{m+1} + C_2\xi^{-m-1} - C_3\xi^{m-1} + (m-2)C_4\xi^{-m+1}$, $f = C_1\xi^m + C_2\xi^{-m}$ $(m \neq 1)$. The constants $C_1, \ldots, C_4$ are found from the boundary conditions (d) = $\psi$(d) = (1) = $\psi'(1) - \psi(1) + im = 0$. We obtain the following for the critical Ma

$$\text{Ma} = \frac{(1-m^2)[m(d^{m-1} - d^{-m-1}) - \text{Bi}(d^{m-1} + d^{-m-1})]}{(1-m^2)G(m, d) + F(m, d)\,\text{We}^{-1}}, \tag{2.7}$$

$$G = \int_d^1 \left(\frac{d^2}{\tau} - \tau\right)\varphi(\tau)[d^{-1-m}\tau^{m+1} + d^{m-1}\tau^{1-m}]\,d\tau,$$

$$F = [(2 - m)(m + 1)C_1 + (2 + m)(1 - m)C_4 - (m + 1)C_2 + (m - 1) \times$$
$$\times C_3 + 1]\,[m(d^m - d^{-m-1})(d^2 - 1) - (d^m + d^{-m-1})(d^2 + 1)].$$

If m = 1, then a nontrivial solution exists only at We = $\infty$ and

$$\text{Ma} = (1 - d^{-2} - (1 + d^{-2})\text{Bi})/G(1, d). \tag{2.8}$$

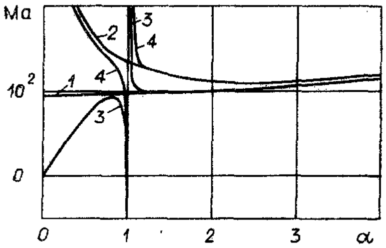


Fig. 3

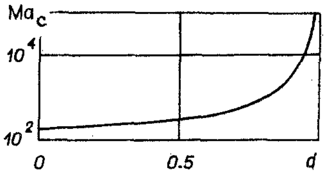


Fig. 4

Here it is also easily shown that Eq. (2.7) changes into (1.19) at $d \to 0$, while the limiting value of (2.8) is 48(1 + Bi).

In the general case $\alpha \neq 0$, $m \neq 0$, we make the substitution (1.21):

$$f = C_1 I_m(\alpha\xi) + C_2 K_m(\alpha\xi),\ \varphi = \frac{C_1}{2}\xi I_m(\alpha\xi) + \frac{C_2}{2}\xi K_m(\alpha\xi) + \frac{C_3}{2} I_{m+1}(\alpha\xi) + $$
$$+ \frac{C_4}{2} K_{m+1}(\alpha\xi) + \frac{C_5}{2} I_{m-1}(\alpha\xi) + \frac{C_6}{2} K_{m-1}(\alpha\xi),$$
$$\psi = -\frac{iC_3}{2} I_{m+1}(\alpha\xi) - \frac{iC_4}{2} K_{m+1}(\alpha\xi) + \frac{iC_5}{2} I_{m-1}(\alpha\xi) + \frac{iC_6}{2} K_{m-1}(\alpha\xi),$$

where the constants $C_1, \ldots, C_6$ are found from the system $\varphi(d) = \psi(d) = \varphi'(d) = \varphi(1) = 0$, $\psi'(1) - \psi(1) + im = 0$, $\varphi''(1) + \varphi'(1) + \alpha^2 + m^2 = 0$.

Following the reasoning in Part 1, we obtain

$$\mathrm{Ma} = \frac{(1-\alpha^2-m^2)(\alpha l_1 + \mathrm{Bi}\, l_2)}{(1-\alpha^2-m^2)G_m + F_m \mathrm{We}^{-1}}, \tag{2.9}$$

where

$$l_1 = I'_m(\alpha d) K_m(\alpha) - K'_m(\alpha d) I_m(\alpha);\ l_2 = K'_m(\alpha) I'_m(\alpha d) - I'_m(\alpha) K'_m(\alpha d);$$
$$G_m = \int_d^1 (d^2 - \tau^2)\varphi(\tau)\left[K_m(\alpha\tau)\ I'_m(\alpha d) - K'_m(\alpha d) I_m(\alpha\tau)\right] d\tau;\ F_m = [-f(1) +$$
$$+ 2\varphi'(1) + 1]\ [\alpha l_1(d^2 - 1) - l_2(d^2 + 1)].$$

The limiting value of Eq. (2.9) at $d \to 0$ coincides with Ma (1.24) for a completely fluid cylinder. As an example, Fig. 3 shows the results of calculation of the critical values of Ma for axisymmetric perturbations by means of Eq. (2.6) with d = 0.2. Curve 1 (Bi = 0, We = ∞) has the minimum $\mathrm{Ma_c} = 93$ at $\alpha_c = 0$, while curve 2 (Bi = 2, We = ∞) reaches the minimum 260 at $\alpha_c = 2.66$. If We ≠ ∞, then the denominator in (2.9) vanishes at $\alpha = \alpha^*$ and $\alpha^* \to 1$, when We → ∞. For We = $10^4$ and $\alpha^* = 1.0014$, the curve 3 (Bi = 0) reaches the maximum 95 at α = 0.88 on the interval $0 \leq \alpha < \alpha^*$, while the minimum $\mathrm{Ma_c} = 100.5$ at α = 1.06. When Bi = 2, then on curve 4 the minimum $\mathrm{Ma_c} = 260$ at $\alpha_c = 2.66$.

With an increase in the wave number (or We), the curves Ma(α, We) rapidly converge and Ma ~ 8α(α + Bi), when We = ∞.

It is evident from a comparison of the curves in Figs. 1 and 3 that asymptotic perturbations of the cylindrical layer are more stable than analogous perturbations of the fluid cylinder. This can be attributed to the stabilizing effect of viscous forces near the solid wall. The curve in Fig. 4, showing the dependence of the minimum critical values of $\mathrm{Ma_c}$ on the parameter d, illustrates the situation being examined $\mathrm{Ma_c}(0) = 178.7$. The same conclusion is valid for m ≥ 1.

3. Plane Layer. The equilibrium state of a plane layer of fluid bounded by a solid lower boundary z = 0 and a free surface z = ℓ is described by the formulas

$$u = v = w = 0,\ p = \text{const},\ \Theta(z) = -qz^2/2\chi + \text{const}. \tag{3.1}$$

As the characteristic temperature, velocity, and pressure, we take the quantities $\nu\ell\gamma/\chi$, $\nu/\ell$, $\rho\nu^2/\ell^2$, where $\gamma = q\ell/\chi$. The boundary-value problem for small perturbations of the equilibrium state (3.1) can be reduced to the following ($\xi = z/\ell$):

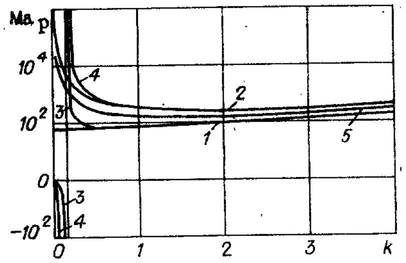


Fig. 5

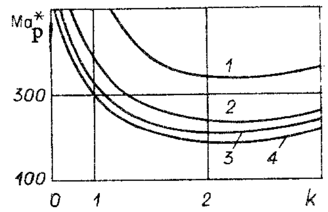


Fig. 6

$$P_{\xi\xi} - k^2 P = 0,\ W_{\xi\xi} - k^2 W = P_\xi,\ T_{\xi\xi} - k^2 T = -zW\ (0 < \xi < 1); \tag{3.2}$$

$$W = W_\xi = T_\xi = 0\ (\xi = 0); \tag{3.3}$$

$$W_{\xi\xi} + k^2\,\mathrm{Ma_p} T + \mathrm{Ma_p}(2W_\xi - P)\,\mathrm{We_p^{-1}} = 0; \tag{3.4}$$

$$T_\xi + \mathrm{Bi_p} T + \frac{\mathrm{Bi_p}+1}{k^2}(2W_\xi - P)\,\mathrm{We_p^{-1}} = 0; \tag{3.5}$$

$$W = 0\ \ (\xi = 1). \tag{3.6}$$

Here $\mathrm{We_p} = \sigma_0 \ell/\rho\nu\chi$; $\mathrm{Ma_p} = \gamma\kappa\ell^2/\rho\nu\chi$; $k^2 = \alpha^2 + \beta^2$; $\alpha$ and $\beta$ are wave numbers in the x and y directions, respectively; k is a wave number.

The solution of problem (3.2) has the form

$$P = C_1 \,\mathrm{sh}\, k\xi + C_2 \,\mathrm{ch}\, k\xi,\ W = \frac{C_1}{2}\left(\xi\,\mathrm{sh}\,k\xi - \frac{\mathrm{ch}\,k\xi}{2k}\right) + \frac{C_2}{2}\left(\xi\,\mathrm{ch}\,k\xi - \frac{\mathrm{sh}\,k\xi}{2k}\right) + C_3\,\mathrm{sh}\,k\xi + C_4\,\mathrm{ch}\,k\xi,$$

$$T = \frac{C_1}{2k}\left(\frac{3}{8k}\xi^2\,\mathrm{sh}\,k\xi - \frac{3}{8k^2}\xi\,\mathrm{ch}\,k\xi + \frac{3}{16k^3}\mathrm{sh}\,k\xi - \frac{1}{6}\xi^3\,\mathrm{ch}\,k\xi\right) + \frac{C_2}{2k}\left(\frac{3}{8k}\xi^2\,\mathrm{ch}\,k\xi - \frac{3}{8k^2}\xi\,\mathrm{sh}\,k\xi + \frac{3}{16k^3}\mathrm{ch}\,k\xi - \frac{1}{6}\xi^3\,\mathrm{sh}\,k\xi\right) +$$
$$+ \frac{C_3}{2k}\left(\frac{1}{2k}\xi\,\mathrm{sh}\,k\xi - \frac{1}{4k^2}\mathrm{ch}\,k\xi - \frac{1}{2}\xi^2\,\mathrm{ch}\,k\xi\right) + \frac{C_4}{2k}\left(\frac{1}{2k}\xi\,\mathrm{ch}\,k\xi - \frac{1}{4k^2}\mathrm{sh}\,k\xi - \frac{1}{2}\xi^2\,\mathrm{sh}\,k\xi\right) + C_5\,\mathrm{sh}\,k\xi + C_6\,\mathrm{ch}\,k\xi.$$

Satisfying boundary conditions (3.3)-(3.6), we obtain

$$\mathrm{Ma_p} = \frac{8k^2 (k - \mathrm{sh}\,k\cdot\mathrm{ch}\,k)(\mathrm{ch}\,k\cdot\mathrm{Bi_p} + k\,\mathrm{sh}\,k)}{\mathrm{sh}^3 k + k^2\,\mathrm{sh}\,k - k\,\mathrm{ch}\,k\cdot\mathrm{sh}^2 k + \frac{2}{3}k^4\,\mathrm{sh}\,k - k^3\,\mathrm{ch}\,k + 8k^3(\mathrm{ch}\,k - k\,\mathrm{sh}\,k)\,\mathrm{We_p^{-1}}}. \tag{3.7}$$

Figure 5 shows the relation $\mathrm{Ma_p}(k)$ calculated from Eq. (3.7). Curve 1 ($\mathrm{Bi_p} = 0$, $\mathrm{We_p} = \infty$) does not have a minimum for any $k > 0$, $\mathrm{Ma_p}(0) = 80$. Along curve 2 ($\mathrm{Bi_p} = 2$, $\mathrm{We_p} = \infty$), the minimum 194.3 is reached at the point $k = 2.18$. If the surface is deformable, then all of the curves $\mathrm{Ma_p}(k)$ have a discontinuity, as in the case of a cylindrical layer or fluid cylinder. Thus, at $\mathrm{We_p} = 10^4$, the discontinuity occurs at the point $k^* = 0.17$. At $0 \le k < k^*$, the critical Marangoni number is nonpositive and $\mathrm{Ma_p}(k) \to -\infty$, $k \to k^* - 0$. Curve 3 ($\mathrm{Bi_p} = 0$, $\mathrm{We_p} = 10^4$) has the minimum 81.2 at $k = 0.59$. For curve 4 ($\mathrm{Bi_p} = 2$, $\mathrm{We_p} = 10^4$), the minimum $\mathrm{Ma_p} = 194.3$, when $k = 2.18$.

The study [1] examined the problem of the development of thermocapillary convection under the influence of a pressure gradient at the boundaries of a layer with $\mathrm{We_p} = \infty$. Curve 5 was taken from this study and corresponds to $\mathrm{Bi_p} = 2$. Since line 2 is located above 5 for all $k > 0$, then the equilibrium state which develops under the influence of constant internal heat sources will be more stable.

It should be noted that at $\mathrm{We_p} = \infty$, all of the cruves (including curve 5) have the same asymptote when $k \to \infty$: $\mathrm{Ma_p}(k) \sim 8k(k + \mathrm{Bi_p})$. As was noted above, the critical Marangoni numbers for the cylinder and cylindrical layer have the same asymptote for short waves. Thus, the critical numbers for shortwave perturbations cease to depend on the type of boundary conditions [3] or equilibrium state or on the dimensions and geometry of the region occupied by the fluid.

It is not hard to show that with $a \to \infty (d \to 1)$ and a fixed value of $b - a = \ell$, the equilibrium state of the cylindrical layer (2.1) reduces to the equilibrium state of the plane layer (3.1). Thus, it is useful to compare the critical Marangoni numbers corresponding to these states. To do this, we need to set

$$\mathrm{Ma}_p^* = 2(1-d^3)\,\mathrm{Ma} \tag{3.8}$$

and to consider that $\alpha = kd/(1-d)$, $m = kd/(1-d)$, $\mathrm{Bi} = \mathrm{Bi}_p/(1-d)$, $\mathrm{We} = \mathrm{We}_p/(1-d)$.

Inserting Ma from Eq. (2.6) or (2.7) into (3.8) and having d approach unity, we obtain the Marangoni number (3.7) in the limit, i.e., $\lim\limits_{d\to 1} \mathrm{Ma}_p^* = \mathrm{Ma}_p$.

Figure 6 shows curves of $\mathrm{Ma}_p^*$ calculated from Eq. (3.8) at $\mathrm{Bi}_p = 2$, $\mathrm{We}_p = \infty$. For curve 1, d = 0.1 and the minimum $\mathrm{Ma}_p = 333.2$, k = 2.23. For curve 2, d = 0.5, $\mathrm{Ma}_p = 237.2$, k = 2.18. For curve 3, d = 0.95, $\mathrm{Ma}_p = 209.2$, k = 1.98. Curve 4 corresponds to the plane layer, when d = 1. It can be seen from Fig. 6 that the minima of the curves for the cylindrical layer are greater than the minimum of the curve for the plane layer. Thus, the equilibrium state of the cylindrical layer is more stable against axisymmetric perturbations than the equilibrium state of the plane layer.

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